Sets & Set Theory

Sets: who cares?

Linguists use sets to provide a semantics for natural language expressions.

Philosophers identify properties and propositions with sets.

Mathematicians study various set theories and believe that many or all mathematical truths could be proved using set theory.

You will use set theory to understand the paper that forms the basis of our final lab report.

What’s a Set?

A **set** is an <u>unordered</u> abstract <u>collection</u> of <u>distinct</u> particular things.

Sets are Collections

Groups

Corporations

Baker’s Dozen

Sets are Unordered

A **sequence** is an *ordered* list of particular things. Whereas sets are distinguished only by their **members** (or **elements**), sequences are distinguished by both which members they contain and the order of those members.

<1, 2, 3, 4, 5> vs <5, 4, 3, 2, 1> ← Distinct **sequences**

{1, 2, 3, 4, 5} vs {5, 4, 3, 2, 1} ← Identical **sets**

Sets Contain Distinct Elements

The elements of a **set** must be distinct from each other. A **set** cannot contain the same thing twice.

{a, u, f, a, h} ← not a set

The elements of a **sequence** do not need to be distinct from each other. A **sequence** can contain the same thing twice.

<a, u, f, a, h> ← a sequence

Which of the following are sets?

Set A: {Taft, Grant, McKinley, Taft, Hoover}

Set B: {Taft, Grant, McKinley, {Taft}, Hoover}

Set C: {Hoover, Grant, Taft, McKinley, {Taft}}

Set D: {Grant, Taft, Hoover, Taft, McKinley}

Set E: {Taft, Set B}

Set F: {<Taft, Grant, McKinley, Taft, Hoover>}

Set G: {Set A}

Which of the following are sets? Answers

Set A: {Taft, Grant, McKinley, Taft, Hoover}	Not a set – repeats
Set B: {Taft, Grant, McKinley, {Taft}, Hoover}	A set
Set C: {Hoover, Grant, Taft, McKinley, {Taft}}	A set
Set D: {Grant, Taft, Hoover, Taft, McKinley}	Not a set – repeats
Set E: {Taft, Set B}	A set
Set F: {<Taft, Grant, McKinley, Taft, Hoover>}	A set
Set G: {Set A}	Debatable

Which of the following are *unique* sets?

Set A: {Taft, Grant, McKinley, Taft, Hoover}

Set B: {Taft, Grant, McKinley, {Taft}, Hoover}

Set C: {Hoover, Grant, Taft, McKinley, {Taft}}

Set D: {Grant, Taft, Hoover, Taft, McKinley}

Set E: {Taft, Set B}

Set F: {<Taft, Grant, McKinley, Taft, Hoover>}

Set G: {Set A}

An item is a *unique set* just in case it is a set and it is distinct from every other set among the items to the left.

Which of the following are *unique* sets? Answers

Set A: {Taft, Grant, McKinley, Taft, Hoover}	Not a unique set – not a set
Set B: {Taft, Grant, McKinley, {Taft}, Hoover}	Not a unique set – identical to Set C
Set C: {Hoover, Grant, Taft, McKinley, {Taft}}	Not a unique set – identical to Set B
Set D: {Grant, Taft, Hoover, Taft, McKinley}	Not a unique set – not a set
Set E: {Taft, Set B}	A unique set
Set F: {<Taft, Grant, McKinley, Taft, Hoover>}	A unique set
Set G: {Set A}	Unique if a set

Abstraction Notation

{1, 7, 44, 72} ← list notation

{x | x is a president of the US} ← abstraction notation

The set of all x such that x is a president of the US

List notation identifies the members of a set by listing them. **Abstraction notation** identifies the members of a set by giving a property which they (and only they) have.

Abstraction Notation Exercise

$\{x \mid 5 < x < 9\} = \{6, 7, 8\}$

$\{x \mid x \text{ is a prime less than } 10\} = \boxed{\qquad\qquad}$

$\{y \mid y \text{ is the room where office hours are held}\} = \boxed{\qquad\qquad}$

$\boxed{\qquad\qquad} = \{1, 3, 5, 7, 9\}$

Abstraction Notation Exercise: Answers

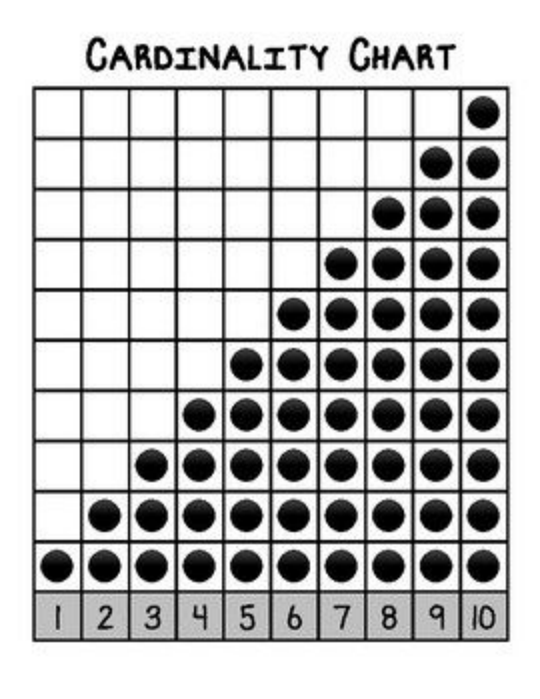
{x | 5 < x < 9} = {6, 7, 8}

{x | the primes less than 10} = {1, 2, 3, 5, 7}

{y | y is the room where office hours are held} = {MHP B5B}

{x | x is odd and <10} = {1, 3, 5, 7, 9}

Cardinality



What is Cardinality?

The **cardinality** of a set is the number of elements in the set.

Notation:

‘The cardinality of x’ is written as ‘|x|’

Examples:

|{Taft, Grant, McKinley, {Taft}, Hoover}| = 5
|{1, 3, 5}| = 3

Cardinality Exercise

|{89, <3, a, b, gh>}| =

|{y | y is the maximum number of students that can register for this section}| =

|{Mckinley, Grant, Obama, {x | x is tree}, Nixon, <{Obama}, {Grant}>} =

Cardinality Exercise: Answers

|{89, <3, a, b, gh>}| = 2

|{y | y is the maximum number of students that can register for this section}| = 1

|{Mckinley, Grant, Obama, {x | x is tree}, Nixon, <{Obama}, {Grant}>} = 6

Cardinality: Special Cases

There is a special set with no members: the **empty set** (or **null set**). We denote the **empty set** with ‘$\varnothing$’ or ‘{ }’. The empty set has a cardinality of 0.

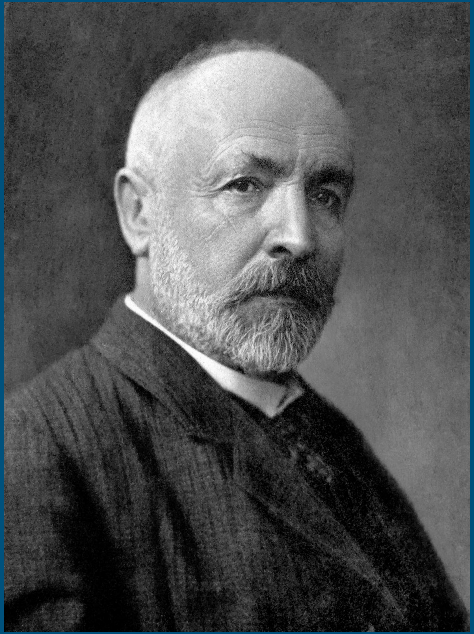
Any set with a cardinality of 1 is called a ‘singleton set’.

Infinite Cardinalities

Cardinalities can be *finite* or *infinite*. Imagine a machine that counts the elements in a set at a constant pace. We give it a set. If it eventually finishes, the set has a *finite cardinality*. If it just keeps counting and never finishes, the set has an *infinite cardinality*. We denote infinite cardinalities with ‘∞’.

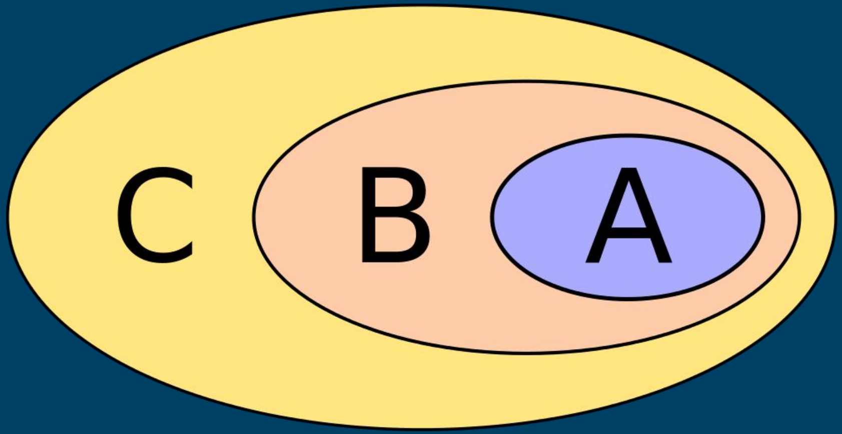
Examples of sets with infinite cardinalities:

{1, 2, 3…}
{x | x is a real number}



Not all sets with infinite cardinalities are the same size…

Set-Theoretic Relations



Membership

A set is a collection of **members** or **elements**. For each element e of a set S, we say that e stands in the **membership** relation to S. We denote this relation between e and S as:

$$e \in S$$

When some particular thing e is not a member of a set S, e does not stand in the **membership** relation to S. We denote this lack of a relation between e and S as:

$$e \notin S$$

Membership Misunderstanding

Membership is not *transitive*. If a member of a set S itself has members, these “grand-members” are not themselves members of S. Thus, a set’s members are only those things it *immediately* contains.

Illustration:

Let A = {a, b}, and B = {A, b}. Then it is true that $a \in A$ and that $A \in B$, but $a \notin B$.

Subset

A is a **subset** of B if and only if every **member** of A is also a member of B.

Notation:

‘A is a **subset** of B’ is written as ‘A $\subseteq$ B’
‘A is not a **subset** of B’ is written as ‘A $\nsubseteq$ B’

Examples:

$\{A\} \subseteq \{A, B\}$
{x | x is an even natural number} $\subseteq$ {x | x is a natural number}
{Trump, McKinley, Taft} $\nsubseteq$ {Taft}

Subset: Special Cases

Every set is a **subset** of *itself*.

Why?

The **empty set** is a subset of *every set*.

Why?

Proper Subset

A is a **proper subset** of B if and only if every member of A is also a member of B, but not every member of B is a member of A. (Equivalently: $A \subseteq B$, but $B \not\subseteq A$.)

Notation:

‘A is a **proper subset** of B’ is written as ‘$A \subset B$’
‘A is not a **proper subset** of B’ is written as ‘$A \not\subset B$’

Examples:

$\{A\} \subset \{A, B\}$
$\{x \mid x \text{ is an even natural number}\} \subset \{x \mid x \text{ is a natural number}\}$
$\{A, B\} \not\subset \{A, B\}$

Relations Exercise: True or False

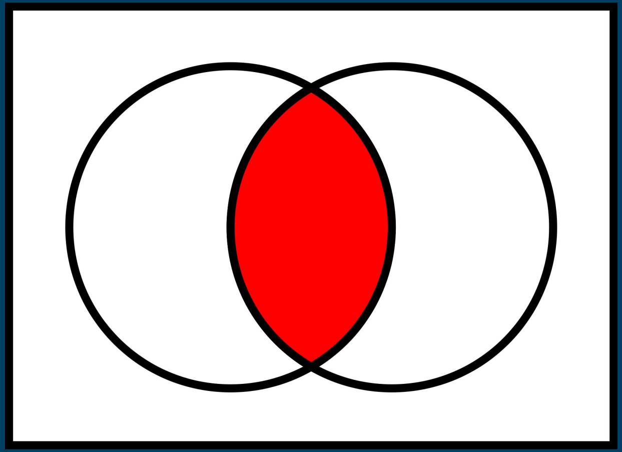
1) $\varnothing \in \varnothing$
2) Taft $\notin$ {{Taft}}
3) $\{A, B\} \in \{B, A\}$
4) $<1, 2, 3> \subseteq <1, 2, 3, 4>$
5) {x | x is a dog} $\subseteq$ {x | x is an animal}
6) $\varnothing \nsubseteq \varnothing$
7) $\{x \mid 1 < x < 3\} \subset \{6, 2\}$
8) $\varnothing \not\subset \varnothing$

Relations Exercise: Answers

1) $\varnothing \in \varnothing$
2) Taft $\notin$ {{Taft}}
3) $\{A, B\} \in \{B, A\}$
4) $<1, 2, 3> \subseteq <1, 2, 3, 4>$
5) {x | x is a dog} $\subseteq$ {x | x is an animal}
6) $\varnothing \nsubseteq \varnothing$
7) $\{x \mid 1 < x < 3\} \subset \{6, 2\}$
8) $\varnothing \not\subset \varnothing$

1) F
2) T
3) F
4) F
5) T
6) F
7) T
8) T

Operations with Sets



Intersection

Take any two sets and compare their members. Pick any members they have in common and make a new set containing all and only those. This new set is the **intersection** of the two original sets.

Notation

‘The intersection of A and B’ is written as ‘A ∩ B’

Examples

$\{1,2,3,4,5\} \cap \{3,4,5,6,7\} = \{3,4,5\}$

$\{x \mid x \text{ is a poodle}\} \cap \{x \mid x \text{ is a dog}\} = \{x \mid x \text{ is a poodle}\}$

$\{\text{Van Buren, Jefferson}\} \cap \{\text{Bush}\} = \varnothing$

The **intersection** of A and B is the set of all things that are members of both A and B.

Subtraction

Take any two sets and compare their members. Pick any members they have in common and make a new set by *removing* those members from the first set. What remains is the **subtraction** of the second set from the first set.

Notation

‘The subtraction of B from A’ or ‘A minus B’ is written as ‘A - B’

Examples

{1,2,3,4,5} - {3,4,5,6,7} = {1,2}
{x | x is a dog} - {x | x is a poodle} = {x | x is a dog and not a poodle}
{Van Buren, Jefferson} - {Bush} = {Van Buren, Jefferson}

The **subtraction** of B from A is the set of all things that members of A but not members of B.

Arithmetic Operations on Cardinalities

Cardinalities are numbers. So we can do arithmetic with cardinalities.

Examples

$|\{\text{Nixon, Ford}\}| + |\{\text{LBJ, JFK}\}| = 4$
$|\{a,z,f\}| - |\{f\}| = 2$
$|\{y \mid y \text{ is a TA for this course}\}| \times |\{y \mid y \text{ is a former or current US president}\}| = 90$

Operations Exercise

$\{x \mid x \text{ is a square}\} \cap \{x \mid x \text{ is a rectangle}\} =$ $\boxed{}$

$\{\{x \mid x \text{ is a square}\}, 7\} \cap \{x \mid x \text{ is a square}\} =$ $\boxed{}$

$\{7, 6, 5, 4\} - \varnothing =$ $\boxed{}$

$\{a, \{ab\}, b\} - \{a\} =$ $\boxed{}$

$|\varnothing| \,/\, |\{x \mid x \text{ is a grain of sand on Venice Beach}\}| =$ $\boxed{}$

$|\{\text{Cleveland, Harding}\}| + |\{\text{Trump}\}| - |\{1,2,3\} \cap \{2\}| =$ $\boxed{}$

Operations Exercise Answers

$\{x \mid x \text{ is a square}\} \cap \{x \mid x \text{ is a rectangle}\} = \boxed{\{x \mid x \text{ is a square}\}}$

$\{\{x \mid x \text{ is a square}\}, 7\} \cap \{x \mid x \text{ is a square}\} = \boxed{\varnothing}$

$\{7, 6, 5, 4\} - \varnothing = \boxed{\{7, 6, 5, 4\}}$

$\{a, \{ab\}, b\} - \{a\} = \boxed{\{\{ab\}, b\}}$

$|\varnothing| \,/\, |\{x \mid x \text{ is a grain of sand on Venice Beach}\}| = \boxed{0}$

$|\{\text{Cleveland}, \text{Harding}\}| + |\{\text{Trump}\}| - |\{1,2,3\} \cap \{2\}| = \boxed{2}$